PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Asymptotics of violent surface motion

S. J. Chapman, K. A. Gillow, S. D. Howison and J. R. Ockendon

Phil. Trans. R. Soc. Lond. A 1997 **355**, 679-685
doi: 10.1098/rsta.1997.0033

Asymptotics of violent surface motion

By S. J. Chapman, K. A. Gillow, S. D. Howison and J. R. Ockendon
OCIAM, The Mathematical Institute, 24–29 St Giles, Oxford OX1 3LB, UK

The very fact that the motion of a free boundary is violent can be of considerable value in analysing the mathematical models of its evolution. For example the smallness of the angle between the undisturbed free surface of a liquid and an impacting smooth solid *almost* allows the problem to be formulated as a classical 'mixed boundary value' problem. This paper gives an informal review of several such configurations and indicates where the violence of the motion enables asymptotic analysis to give new insight, both qualitatively and quantitatively. However, many of these situations are in practice irreversible and we will explain why this can cause severe difficulties with the mathematics.

1. Introduction

Both the theoretical and the numerical analysis of many violent surface motions, for example those generated by impacts, are difficult. On the theoretical side, the presence of a free surface with its concomitant nonlinearity usually makes the study of existence and uniqueness hard, and renders it impossible to find explicit solutions in all bar a few cases. Numerical analysis is in addition severely hampered by the rapid motion of the free surface and by the presence of small regions within which large changes occur. However, both these latter circumstances favour an asymptotic approach, and in this paper we describe how this may, in certain circumstances, be carried out. Our intention is to give a brief review of some mathematical aspects of these problems, and to make some conjectures concerning open questions raised thereby.

We shall mainly discuss water-entry and water-exit of blunt bodies, as a canonical example of the type of problem where asymptotic analysis is effective, and where it can be confirmed by more rigorous analysis (Fraenkel, this volume). We note, though, that there is a great variety of problems to which the methodology is applicable, and refer to the reviews by Howison (1991), Howison *et al.* (1997) and Ockendon (1991) for further examples and more details.

The basis for asymptotic approximation is that in a violent impact such as water-entry of a smooth body, the lateral extent of the 'contact set' between the impacting body and the water is much greater than the depth of penetration, a fact which follows from simple geometric considerations (see figure 1). One can then linearize the boundary conditions on both the free surface and the impacting body onto the undisturbed water level, giving a mixed boundary value problem in a domain which is known except for the points demarcating the contact set. A careful consideration of the conditions that apply at these points may then allow the determination not

Phil. Trans. R. Soc. Lond. A (1997) **355**, 679–685
Printed in Great Britain

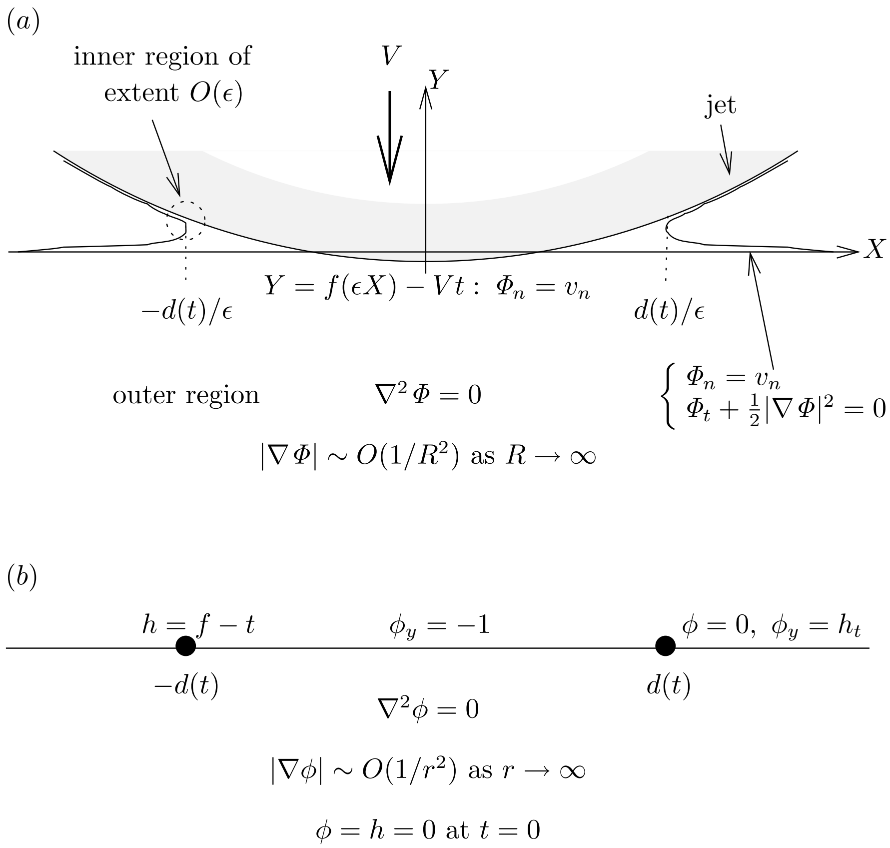

Figure 1. The water-entry problem and its linearized version.

only of the relevant physical quantities (here, the velocity and pressure) but also of the extent of the contact set.

2. Water-entry

The simplest possible model of the water-entry of a blunt symmetric two-dimensional body $Y = f(\epsilon X)$ is shown in figure 1*a*. The fluid motion is modelled by incompressible irrotational flow, with velocity potential $\Phi(X, Y, t)$; the kinematic and dynamic conditions on the free surface, and the kinematic condition on the body are as shown. Thus we have neglected, among others, effects of gravity, viscosity, three-dimensionality, compressibility, surface tension and flow in the air above the water; some of these effects are considered elsewhere in this volume (Cointe, this volume; Greenhow, this volume; Korobkin, this volume).

A detailed examination (Howison *et al.* 1997) of the asymptotic structure (confirmed rigorously in Fraenkel (this volume) when the impacting body is a wedge) of the solution as $\epsilon \to 0$ shows that the three types of region shown in figure 1*a* play an important role. First is an 'outer' region in which the fluid response is as if the body were an expanding flat plate between the points $X = \pm d(t)/\epsilon$ moving with velocity $(0, -V)$: this is the scenario first proposed in Wagner (1932). Then in the two 'turn-around' regions of size $O(\epsilon)$ adjacent to the body near $X = \pm d(t)/\epsilon$, we have a travelling-wave free surface flow of Helmholtz type; and finally the water is ejected as two thin jets emanating from the turn-around regions.

It is now possible, by neglecting the jets (which can be justified *a posteriori*) to linearize the conditions on the free surface onto the X-axis. Then after rescaling $(X, Y) = (x, y)/\epsilon$, $\Phi = \phi/\epsilon$, the free surface displacement is $y = \epsilon h(x, t)$ and the body is at $y = \epsilon(f(x) - t)$. After linearization and expansion in powers of ϵ the resulting mixed boundary value problem for the leading-order velocity potential, which without risk of confusion we denote by $\phi(x, y, t)$ (with a similar convention for $h(x, t)$ and $d(t)$) is as shown in figure 1*b*. For any $d(t)$ we can write down ϕ by inspection:

$$\phi = -y - \mathrm{Re}\{\sqrt{d(t)^2 - (x + \mathrm{i}y)^2}\}.$$

In order to determine $d(t)$, we use the fact that the free surface only 'turns over' when it is very close to the body (in the scaled coordinates the extent of this turnover region is $O(\epsilon^2)$). Thus we impose the condition

$$h(d(t), t) = f(d(t)) - t,$$

and, integrating the linearized kinematic boundary condition $\phi_y = h_t$, we obtain the integral equation

$$f(d(t)) = \int_0^t \frac{d(t)}{(d(t)^2 - d(\tau)^2)^{1/2}}\,\mathrm{d}\tau,$$

the solution of which is (Cointe & Armand 1987; Howison *et al.* 1991; Tollmien 1934; Wagner 1932)

$$d^{-1}(x) = \frac{2}{\pi}\int_0^x \frac{f(\zeta)}{(x^2 - \zeta^2)^{1/2}}\,\mathrm{d}\zeta.$$

As stated above, these formal results are confirmed when $f(x) = |x|$, in which case $d(t) = \pi t/2$ (Fraenkel, this volume).

The simplest model outlined previously has some interesting generalizations and consequences. For example, it is easy to allow the initial water surface to be non-planar; thus for example the model can be used to predict jet formation when a smoothly breaking wave violently impacts a vertical breakwater (Cooker & Peregrine (1995) present numerical calculations of this situation). We also note that the analysis may be complicated by the possibility of a 'critical' jet (see Longuet-Higgins, this volume). The formulation in three dimensions follows similar lines, although explicit solutions are only available in special cases (see Howison *et al.* (1991) for a catalogue). Oblique impact (in which the impacting body has a transverse velocity as well as a normal velocity) leads to the same model and, if the body is symmetric, gives a splash that is symmetric to leading order unless the transverse velocity is at least of $O(V/\epsilon)$ (Morgan 1994).

Of more mathematical interest is the fact that an integration in time leads to a reformulation as a variational inequality for the 'displacement potential' (Korobkin 1982). This in turn allows existence and uniqueness to be established for the 'linearized' problem in figure 1*b*, and is an effective formulation for numerical computation (Howison *et al.* (1991) reports comparison with experiment).

3. 'Codimension-two' free boundary problem

The linearization procedure described previously has been applied in a wide variety of physical situations, some of which are listed later and others of which are reviewed in Howison (1991), Howison *et al.* (1997) and Morgan (1994). In all cases the general procedure is the same.

(1) Assume that the free surface in the 'full' problem lies close to a known surface.

(2) Linearize the free surface boundary conditions onto this known surface. (This procedure may also be carried out for conditions applied on a prescribed surface; in the water-entry problem the conditions on the body fall into this category.) Then the only geometrical unknown is the curve bounding the projection of the free surface onto the known surface; because this has codimension two, we term it the 'codimension-two free boundary'.

(3) Solve the resulting mixed boundary value problem and determine the codimension-two free boundary by applying appropriate conditions there. These may come from a matched asymptotic expansion argument in which the linearized free boundary is matched to a local solution of the full problem (as in water-entry); they may alternatively come from minimum singularity arguments.

Examples of the procedure include:

(i) contact problems and crack problems in linear elasticity (here the linearization is implicit in the formulation via linear elasticity);

(ii) flow over a shallow step (O'Malley *et al.* 1991);

(iii) patch cavitation on a bluff headform (Ceccio & Brennen 1991; Howison *et al.* 1997);

(iv) evolution of long thin bubbles in Hele-Shaw flow (Hohlov *et al.* 1994; Lacey *et al.* 1990) and Stokes flow; and

(v) the initial stages of the sintering of two cylinders of viscous liquid under the action of surface tension (Hopper 1992; Howison *et al.* 1997; Morgan 1994).

In the last two cases, exact solutions to the full problem are available to confirm the validity of the approximation.

4. Water-exit problems

As indicated elsewhere in this volume, water-exit problems are considerably more problematical than entry problems. Indeed, from the physical and mathematical points of view, it is neither clear how to pose the problem of rapid water-exit in sensible terms, nor whether existence or, even less likely, uniqueness, are to be expected. As examples of the mathematical difficulties we note the following.

(1) The full problem (as stated in figure 1*a*) is formally time-reversible. Thus one can generate solutions for the exit of a partially submerged body by first considering the entry of the same body into undisturbed water (or indeed disturbed water, i.e. water that is in motion or does not initially have a planar free surface), then reversing time. However, it is not clear how large is the set of initial conditions for which this procedure generates physically acceptable solutions.

(2) Another conceivable solution to the exit of a partially submerged body from (say) initially static water is that the body is simply removed, leaving the water motionless (in the absence of surface tension). Of course this solution demands that contact between the body and the water is immediately lost on all the wetted portion of the body, which may be physically unrealistic.

Turning to the linearized problem, the solution procedure outlined here fails, simply because whereas in entry problems, for each x the time $d^{-1}(x)$ is a natural *terminus ad quem* for integration of the equation $\phi_y = h_t$, this relies on the fact that $d(t)$ is increasing. For exit problems, the converse is true, and it is possible to construct infinitely many solutions to the exit problem (figure 1*a* with $\phi_y = +1$ for $-d(t) < x < d(t)$, $y = 0$). Note, though, that almost all of these leave the water surface with a 'kink' at $x = d(0)$, and may indeed blow up in finite time (a similar situation arises for Hele-Shaw suction problems compared to injection problems (Lacey 1990)).

A final pointer to the ill-posedness of the exit problem is a local linear stability analysis of the codimension-two free boundary problem. This analysis is somewhat unconventional: we have to determine the stability of the line $x = d(t)$, $y = 0$, $-\infty < z < \infty$ (where z is the third Cartesian coordinate) to perturbations in the z-direction but constrained to lie in the x–y plane, so that the perturbed codimension-two free boundary is $x = d(t) + \delta\sin(nz)d_1(t)$, $y = 0$, $-\infty < z < \infty$, where $\delta \ll 1$, $n > 0$ and $d_1(t)$ is to be determined. The problem is otherwise as stated in figure 1*b*, except that in the matching condition $h(d(t), t) = f(d(t)) - t$, $d(t)$ is replaced by $d(t) + \delta\sin(nz)d_1(t)$. It is necessary to construct an inner region near the perturbed codimension-two free boundary, of size $O(\delta)$; the potential is calculated to two orders in δ in both this region and the outer region, and after considerable manipulation (space does not permit a full treatment here), it is found that the growth rate $d_1(t)$ satisfies

$$d_1(t) = \text{const.}\,\frac{\dot{d}(t)\mathrm{e}^{-nd(t)}}{A(t)},$$

where $A(t)$ is a globally determined function. This clearly demonstrates stability for entry ($V > 0$, $d(t)$ increasing) and instability for the time-reversal (exit) problem ($V < 0$, $d(t)$ decreasing).

It is interesting to compare this analysis with the linear stability analyses of (*a*) conventional 'codimension-one' free boundary problems, for example surface gravity waves or the Kelvin-Helmholtz analysis of a vortex sheet and (*b*) the motion of a free singular curve such as a line vortex in inviscid hydrodynamics (which is also a codimension-two free boundary problem, but without the constraint that the free curve must lie on a prescribed surface). In the former case there is no singular behaviour near the free boundary, and the analysis is straightforward without the need for matched asymptotic expansions. In the latter, the behaviour is singular and matched asymptotic expansions are necessary; the upshot is that both the singular behaviour and the 'law of motion' are, to leading order, *locally* determined. Thus for a vortex, the local behaviour in the fluid velocity is $\boldsymbol{u} \sim \text{const.}\,\boldsymbol{e}_\theta/r$ with respect to cylindrical polars aligned with the tangent to the vortex, and the evolution is given by $\boldsymbol{v} = \text{const.}\,\boldsymbol{b}\kappa$ where κ is the curvature, $\boldsymbol{b}$ the binormal and $\boldsymbol{v}$ the normal velocity of the vortex. In our case, the singular behaviour is still *locally* determined (and is thought of as matching with the local form of the full, codimension-one, problem), but the 'law of motion', namely

$$h(d(t) + \delta\sin(nz)d_1(t), t) = f(d(t) + \delta\sin(nz)d_1(t)) - t,$$

is *globally* determined because $h(x, t)$ is found by a global integration in t.

Finally we remark in connection with exit problems that an ingenious solution has been developed by the Basilisk Lizard, *Basiliscus basiliscus* (Glasheen & McMahon 1996). These reptiles can run on water, and support their weight by exploiting the large pressures generated when their feet hit the water surface approximately normally. Their feet subsequently penetrate some distance into the water, creating a cavity (and are hence well into the full nonlinear regime), and are then rotated so that they can be withdrawn more or less vertically through the cavity before it collapses. Large negative pressures, which would act to suck the lizard down, are thereby avoided.

5. Discussion

To conclude, we mention some unresolved issues, to do with both water-entry and water-exit problems. Taking the latter first, the need for some regularization is apparent. On physical grounds, one might propose the incorporation of other effects into the model; for example, on small length scales surface tension may be a significant smoothing mechanism, while in other circumstances a coupled model involving air moving underneath the exiting body may be more appropriate. A second possibility is regularization via a weak solution of the kind described in Rogers (this volume), allowing a partially saturated region to form underneath the body; this idea is appealing both in the light of observed instabilities under exiting cylinders (Greenhow, this volume) and in view of the instability described in § 4. Indeed, one might think of a partially saturated region either as being due to cavitation or as a model for the effect of a large number of thin fingers of air that have developed from an initially small perturbation; this idea has much in common with models of 'mushy regions' in Stefan problems with volumetric heating (Lacey & Tayler 1983).

Finally we mention a purely mathematical possibility: that the solution to the exit problem should be the time-reversal of an entry problem, subject to the condition that the free surface (in the full problem), or $h(x,t)$ in the linearized problem, should be uniformly smooth. However, there is so far no evidence to support this hypothesis.

Turning to entry problems, we note some possible difficulties with liquid–liquid impacts. As noted in Howison *et al.* (1991), the codimension-two approximation can in principle also be used here. However, consideration of the evolution of the flow immediately after first contact suggests that there are several different possible scenarios, each requiring a different modelling approach, and likely to occur in different parameter regimes. Among these are the following.

(1) The two fluid masses immediately cohere and the distinction between them is lost; this is the approach of Blake (this volume) and Prosperetti (this volume). The velocity is then smooth in the interior of the flow domain.

(2) A vortex sheet separating the two liquids may in certain circumstances be appropriate. This possibility is explored in more detail in Cresswell & Morton (1995) and Zhang *et al.* (1993).

(3) For large-scale flows, air may be entrained: note, though, that contact does eventually occur, albeit at a lower relative velocity.

(4) Because the curvature of the surface at contact is instantaneously infinite, surface tension effects must be significant albeit only locally. If, however, the impacting velocity is small or zero, as would occur in the inviscid version of the sintering problem mentioned previously or in low-velocity drop impact, the effects of surface tension may be more significant. Hitherto attempts to formulate the short-time behaviour of 'inviscid sintering' have not succeeded; however, we note that Prosperetti & Oğuz (1989) raise the intriguing possibility that capillary wave propagation along the free surface very quickly leads to another kind of 'mushy region' involving multiple contact with trapped air bubbles. The detailed modelling of this phenomenon, and the assessment of its effect on the later development of the flow, remains an interesting open problem.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

S.J.C. is grateful for support from the Royal Society via a University Research Fellowship, and K.A.G holds an EPSRC research studentship.

References

Ceccio, S. L. & Brennen, C. E. 1991 Observations of the dynamics and acoustics of travelling bubble cavitation. *J. Fluid Mech.* **233**, 633–660.

Cooker, M. J. & Peregrine, D. H. 1995 Pressure-impulse theory for liquid impact problems. *J. Fluid Mech.* **297**, 193–214.

Cointe, R. & Armand, J. L. 1987 Hydrodynamic impact analysis of a cylinder. *J. Off. Mech. Arc. Engng* **107**, 237–243.

Cresswell, R. W. & Morton, B. R. 1995 Drop-formed vortex rings – the generation of vorticity. *Phys. Fluids* **7**, 1363–1370.

Glasheen, J. W. & McMahon, T. A. 1996 A hydrodynamic model of locomotion in the Basilisk lizard. *Nature* **380**, 340–342.

Hohlov, Y. E., Howison, S. D., Huntingford, C. & Ockendon, J. R. 1994 A model for non-smooth free boundaries in Hele-Shaw flow. *Q. J. Mech. Appl. Math.* **47**, 107–128.

Hopper, R. W. 1992 Stokes flow of a cylinder and half-space driven by capillarity. *J. Fluid Mech.* **243**, 171–181.

Howison, S. D. 1991 Codimension-two free boundary problems. In *Proc. Int. Coll. on Free Boundary Problems* (ed. J. Chadam & H. Rasmussen). London: Pitman.

Howison, S. D., Ockendon, J. R. & Wilson, S. K. 1991 Incompressible water-entry problems at small deadrise angle. *J. Fluid Mech.* **222**, 215–230.

Howison, S. D., Morgan, J. D. & Ockendon, J. R. 1997 Codimension-two free boundary problems. *SIAM Rev.* (To appear.)

Korobkin, A. A. 1982 Formulation of penetration problem as a variational inequality. *Din. Sploshnoi Sredy* **20**, 73–79.

Lacey, A. A., Howison, S. D., Ockendon, J. R. & Wilmott, P. 1990 Irregular morphologies in unstable Hele-Shaw free-boundary problems. *Q. J. Mech. Appl. Math.* **43**, 387–405.

Lacey, A. A. & Tayler, A. B. 1983 A mushy region in a Stefan problem. *IMA J. Appl. Math.* **30**, 303–313.

Morgan, J. D. 1994 Codimension-two free boundary problems. D.Phil. thesis, Oxford University.

Ockendon, J. R. 1991 A class of moving boundary problems arising in industry. In *Proc. Venice Conf. Appl. Ind. Math.* Dordrecht: Kluwer.

O'Malley, K., Fitt, A. D., Jones, T. V., Ockendon, J. R. & Wilmott, P. 1991 Models for high Reynolds number flow down a step. *J. Fluid Mech.* **222**, 139–155.

Prosperetti, A. & Oğuz, H. N. 1989 Surface-tension effects in the contact of liquid surfaces. *J. Fluid Mech.* **203**, 149–171.

Tollmien, W. 1934 Zum landestoß von seeflugzeugen. *ZAMM* **14**, 251.

Wagner, H. 1932 Über stoß- und gleitvorgänge an der oberfläche von flüs-sigkeiten. *ZAMM* **12**, 193–215.

Zhang, S., Duncan, J. H. & Chahine, G. L. 1993 The final stage of the collapse of a cavitation bubble near a rigid wall. *J. Fluid Mech.* **257**, 147–181.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES